Föreläsning 21/11-13

Lemma 6.3.5
If the state space S is finite (= finite number of possible values for Markov chain). Then at least one state is persistent and all persistent states are non-null.

Proof: $1 = \sum_j P_{ij}(n)$

If j is transient then $\sum_{n=1}^{\infty} P_{ij}(n) < \infty \Rightarrow P_{ij}(n) \to 0$ as $n \to \infty$

If all j is transient then $P_{ij}(n) \to 0$ as $n \to \infty$ for every j.

$\lim_{n\to\infty} 1 = \lim_{n\to\infty} \sum_j P_{ij}(n) = \sum_j \lim_{n\to\infty} P_{ij}(n) = 0$ contradiction

$\Rightarrow$ at least one j is persistent. ▨

$S = T \cup C_1 \cup C_2 \ldots$

T transient states

C_1 closed set of persistent states that intercommunicate

C_2 ——— || ———

$\vdots$

Thm 6.3.2 in each C_n all states are either null or non-null.
Assume that one of the C's, say C_n, have null states,
$\mu_j = \infty$ all $j \in C_n$
Thm 6.2.9 says that $P_{ij}(k) \to 0$ as $k \to \infty$ $\forall j \in C$.

$$1 = \sum_{\substack{\text{all } j \\ (i \text{ in } C_n)}} P_{ij}(k) = \sum_{j \in C_n} P_{ij}(k)$$

$\lim_{k\to\infty} 1 = \lim_{k\to\infty} \sum_{j\in C_n} P_{ij}(k) = \sum_{j\in C_n} \lim_{k\to\infty} P_{ij}(k) = 0$ contradiction as before. ▨

6.5 Reversibility

X_n Markov chain, started according to it's stationary dist. (which is assumed to exist).

Consider time reversed process $Y_n = X_{N-n}$ for some (big) N.

$X_n \xrightarrow{n}$ N
$Y_n \xleftarrow[n]{}$

- Is Y_n also a Markov chain?

Check: $P(Y_{n+1} = i_{n+1} \mid Y_n = i_n, \ldots, Y_0 = i_0) = \dfrac{P(Y_{n+1} = i_{n+1}, Y_n = i_n, \ldots, Y_0 = i_0)}{P(Y_n = i_n, \ldots, Y_0 = i_0)} =$

$$= \frac{P(X_N = i_0, X_{N-1} = i_1, \ldots, X_{N-n-1} = i_{n+1})}{P(X_N = i_0, X_{n-1} = i_1, \ldots, X_{N-n} = i_n)} = \frac{\mu^{(N-n-1)}_{i_{n+1}} P_{i_{n+1} i_n} \cdots P_{i_2 i_1} P_{i_1 i_0}}{\mu^{(N-n)}_{i_n} P_{i_n i_{n-1}} \cdots P_{i_2 i_1} P_{i_1 i_0}} =$$

$$= \frac{\pi_{i_{n+1}} P_{i_{n+1} i_n}}{\pi_{i_n}} \Rightarrow P(Y_{n+1} = j \mid Y_n = i, \ldots) = \frac{\pi_j P_{ji}}{\pi_i} \Rightarrow \text{Markov!}$$

• When does Y_n have same transition matrix as X_n?
def. when that is so we call X_n reversible!

Check: Happens when $p_{ij} = \frac{\pi_j P_{ji}}{\pi_i} \iff \pi_i P_{ij} = \pi_j P_{ji}$ all i and j.

Thm
Suppose we have a row matrix π which is a PMF such that $\pi_i P_{ij} = \pi_j P_{ji}$ all i, j. The π is stationary distribution for X and Y is reversible.

Proof: Prove that $\pi P = \pi$, $(\pi P)_k = \sum_j \pi_j P_{jk} = \left(\sum_j \pi_k P_{kj}\right) \overset{=1}{=} \pi_k$ ▨

Is all about continous Markov chains.
$P(X_{t_{n+1}} = x_{n+1} \mid X_{t_n} = x_n, \ldots, X_{t_0} = x_0) = P(X_{t_{n+1}} = x_n \mid X_{t_n} = x_n)$
for discrete valued coutinous time process $(X_t ; t \geq 0)$.